*International Journal of Theoretical Physics*, Vol. 6, No. 4 (1972), pp. 293–299

# The Group of Automorphisms of the Galilei Group

G. MARMO

*Istituto di Fisica Teorica, Università di Napoli*

and

G. S. WHISTON†

*Istituto Nazionale di Fisica Nucleare, Sezione di Napoli*

*Received: 26 October 1971*

*Abstract*

The group of automorphisms of the Galilei group $G$: Aut($G$) is calculated. It is shown that Aut($G$) has the structure of a semi-direct product by $G$ of the group $\mathbb{R}_m^* \times \mathbb{R}_m$ where $\mathbb{R}_m$ is the group of reals noted multiplicatively and $\mathbb{R}_m^* < \mathbb{R}_m$ is the subgroup of positive reals.

## *Introduction*

There is a threefold motivation for a calculation of the group Aut($G$) of abstract automorphisms of the Galilei group $G$. Firstly, one may wish to compare Aut($G$) with the group of automorphisms of the Poincaré group $\mathbb{P}$.Aut($\mathbb{P}$) (Michel, 1967). Secondly, one may be interested in computing group extensions of an 'internal' group by $G$. In this case one needs to know the algebraic structure of Aut($G$) to discuss the extensibility of certain '$Q$-kernels' (Michel, 1966). Thirdly, the result is of course of considerable interest in itself. This article will adopt the third point of view, leaving possible applications to a later paper. $G$ is of very great interest in its own right (Lévy-Leblond, 1971).

In the calculation, we shall use methods based on those used by Michel (1967) in his calculation of Aut($\mathbb{P}$). The present calculation, although longer than the latter, is simpler in essence since the more complex algebraic

† Current address: Department of Mathematics, University of Durham, England.

structure of $G$ (Whiston, 1972) imposes more conditions on any automorphism of **G**.

The scheme of this paper is as follows. We first state, without proof, a theorem (proved by Michel (1967)) on the automorphisms of any group **G** which is a semi-direct product by a characteristic kernel, and a theorem on the automorphisms of the full rotation group $O(3,\mathbb{R})$. These results are used in order to establish a Lemma on the structure of the automorphism group of the three-dimensional Euclidian group which is needed for the final calculation of $\mathrm{Aut}(G)$.

*Lemma* (1)

Suppose $\mathbf{G} = KXpQ$ where $K$ is a characteristic subgroup of **G**, $K \lhd \mathbf{G}$, and $K$ is defined as a $Q$-module via the homomorphism $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$. Any element $f \in \mathrm{Aut}(\mathbf{G})$ sends the element $(k,q)$ of **G** into the element $(\alpha_f(k) + \beta_f(q), \gamma_f(q))$ where

(a) $\alpha_f \in \mathrm{Aut}(K)$
(b) $\gamma_f \in \mathrm{Aut}(Q)$
(c) $\alpha_f \in \mathrm{Hom}((K,p),(K,p \circ \gamma))$
(d) $\beta_f \in Z^1_{p \circ \gamma}(Q,K)$

Equation (c) means that $\alpha$ is a module homomorphism between the $Q$-modules $(K,p)$ and $(K,p \circ \gamma)$ and (d) means that $\beta_f$ is a one cocycle of $Q$ in the $Q$-module $(K,p \circ \gamma)$.† If we note that $O(3,\mathbb{R}) \cong \mathbb{Z}_2(p) \times SO(3,\mathbb{R})$ (where $P$ is the parity operator and $\mathbb{Z}_2(g)$ is the two-element cyclic group generated by $(g)$ and $SO(3,\mathbb{R})$ is complete.

Then an extension of Lemma (1) with $Q \lhd G$ implies

*Lemma* (2)

$$\mathrm{Aut}(O(3,\mathbb{R}) \cong \mathrm{Int}(SO(3,\mathbb{R})) \cong SO(3,\mathbb{R})$$

These first two Lemmas will be used to establish Lemma (3) below.

*Lemma* (3)

Any automorphism of the three-dimensional Euclidean group

$$E(3,\mathbb{R}) \cong \mathbb{R}^3 \, X_n \, O(3,\mathbb{R})$$

(where '$n$' is the natural module action of $O(3,\mathbb{R})$ on $\mathbb{R}^3$) sends the element $(\mathbf{v},R)$ of $E(3,\mathbb{R})$ to the element:

$$(\delta_f R_f \mathbf{v} + \mathbf{k}_f - R_f R R_f^{-1} \mathbf{k}_f, R_f R R_f^{-1})$$

† See either Michel 1966 or 1967 for an elementary exposition of the cohomology theory of abstract groups.

where $\delta_f$ is a non-zero real number, $\mathbf{k}_f$ a vector of $\mathbb{R}^3$ and $R_f$ is a proper rotation. Equivalently

$$\mathrm{Aut}(E(3,\mathbb{R})) \cong E(3,\mathbb{R})\, X_\theta\, \mathbb{R}_m^*$$

where $\theta \in \mathrm{Hom}(\mathbb{R}_m^*, \mathrm{Aut}(E(3,\mathbb{R}))$ is given by $\theta(\delta)(\mathbf{v}, R) \equiv (\delta\mathbf{v}, R)$.

*Proof*

Certainly $\mathbb{R}^3 \lhd E(3,\mathbb{R})$ so we may use Lemma (1) to write for any $f \in \mathrm{Aut}(E(3,\mathbb{R}))$

$$f\colon (\mathbf{v}, R) \mapsto (\alpha_f(\mathbf{v}) + \beta_f(R), \gamma_f(R))$$

where $\alpha_f$ is an automorphism of $\mathbb{R}^3$, $\gamma_f$ is an automorphism of $O(3,\mathbb{R})$ and

$$\beta_f \in Z^1_{n \circ \gamma_f}(O(3,\mathbb{R}), \mathbb{R}^3)$$

By Lemma (2) $\gamma_f$ is the inner automorphism:

$$\gamma_f\colon R \mapsto R_f R R_f^{-1},\ R_f \in SO(3,\mathbb{R})$$

Therefore the condition on $\alpha_f$ that $\alpha_f$ be a module homomorphism is that

$$\alpha_f(R\mathbf{v}) = R_f R R_f^{-1} \alpha_f(\mathbf{v}) \qquad \text{for any } R \in O(3,\mathbb{R})$$

Let $I(\mathbf{v})$ denote the isotropy group of $\mathbf{v}$ in $O(3,\mathbb{R})$. Then the latter equation is

$$I(\alpha_f(\mathbf{v})) = I(R_f \mathbf{v})$$

But two vectors have the same little group iff they are colinear (Michel, 1967). Consequently, the module condition gives us that

$$\alpha_f(\mathbf{v}) = \delta_f R_f \mathbf{v}$$

where $\delta_f$ is a non-zero real number. (Since $\alpha_f$ is an automorphism.) The calculation of $\beta_f$ is also simple. We have the condition

$$\beta_f(R_1 R_2) = \beta_f(R_1) + R_f R_1 R_f^{-1} \beta_f(R_2)$$

which, together with the fact that the centre of $O(3,\mathbb{R})$ is $\mathbb{Z}_2(P)$, means that

$$\beta_f(R) = \mathbf{k}_f - R_f R R_f^{-1} \mathbf{k}_f$$

where $\mathbf{k}_f = \beta_f(P)/2$ is a vector of $\mathbb{R}^3$. Consequently, we have shown that

$$f\colon (\mathbf{v}, R) \to (\delta_f R_f \mathbf{v} + \mathbf{k}_f - R_f R R_f^{-1} \mathbf{k}_f, R_f R R_f^{-1})$$

where $R_f \in SO(3,\mathbb{R})$, $\delta_f \in \mathbb{R}$, $\delta_f \neq 0$ and $\mathbf{k}_f \in \mathbb{R}^3$. If we apply the last result twice, the correspondence $f \mapsto ((k_f, R_f), \delta_f)$ is an isomorphism

$$\mathrm{Aut}(E(3,R)) \cong E(3,\mathbb{R})\, X_\theta\, \mathbb{R}_m^*$$

This is immediately applicable to the calculation of the automorphisms of $G$.

*Theorem*

Any automorphism of the Galilei group $G$ sends the element $((\mathbf{x},t),(\mathbf{v},R))$ of $G$ into the element

$$((\delta_f \sigma_f R_f \mathbf{x} + \sigma_f \mathbf{k}_f t - \tau_f R_f \mathbf{v} + \mathbf{X}_f - R_f R R_f^{-1} \mathbf{X}_f, \sigma_f t),$$
$$(\delta_f R_f \mathbf{v} + k_f - R_f R R_f^{-1} k_f, R_f R R_f^{-1}))$$

where $\delta_f$, $\delta_f \neq 0$ are real numbers $\tau_f$ is a real number $X_f$, $R_f$ are vectors of $\mathbb{R}^3$ and $R_f$ is a proper rotation. Equivalently

$$\mathrm{Aut}(G) \cong G x_\omega(\mathbb{R}_m^* \times \mathbb{R}_m)$$

where the action of $\mathbb{R}_m^* \times \mathbb{R}_m$ as a group of automorphisms of $G$ is given by

$$\omega(\delta,\sigma)\colon ((\mathbf{x},t),(\mathbf{v},R)) \mapsto ((\delta\sigma\mathbf{x},\sigma t),(\delta\mathbf{v},R))$$

*Proof*

It is clear that $\mathbb{R}^3 \times \mathbb{R} \lhd G \cong (\mathbb{R}^3 \times \mathbb{R}) X_\rho E(3,\mathbb{R})$ where $\rho \in \mathrm{Hom}(E(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3 \times \mathbb{R}))$ is given by

$$\rho(\mathbf{v},R)\colon ((\mathbf{x},t) \mapsto (R\mathbf{x}+\mathbf{v}t,t))$$

(this follows because $\mathbb{R}^3 \times \mathbb{R}$ is the only four-dimensional abelian invariant subgroup). We may therefore apply Lemma (1) to write for any $f \in \mathrm{Aut}(G)$

$$f\colon ((\mathbf{x},t),(\mathbf{v},R)) \mapsto (\alpha_f(\mathbf{x},t)\,\beta_f(\mathbf{v},R), \gamma_f(\mathbf{v},R))$$

where $\gamma_f$ is an automorphism of $E(3,\mathbb{R})$, $\alpha_f$ is an automorphism of $\mathbb{R}^3 \times \mathbb{R}$ and a module homomorphism and $\beta_f$ is a one cocycle of the group $E(3,R)$ in the module $(\mathbb{R}^3 \times \mathbb{R}, \rho \circ \gamma_f)$. We first calculate $\alpha_f$. It can be shown (Michel, 1967) that $\alpha_f$ is $\mathbb{R}$-linear. The module condition on it is:

$$\alpha_f(R\mathbf{x}+\mathbf{v}t,t) = \rho \circ \gamma_f(\mathbf{v},R)\,(\alpha_f(\mathbf{x},t))$$

Let us write:

$$\alpha_f\colon (\mathbf{x},t) \mapsto (\alpha_1(\mathbf{x},t), \alpha_2(\mathbf{x},t))$$

Then $\alpha_1$ and $\alpha_2$ are $\mathbb{R}$-linear maps $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \mathbb{R}$ respectively and we have

$$\alpha_2(R\mathbf{x}+\mathbf{v}t,t) = \alpha_2(\mathbf{x},t)$$

$$\alpha_1(R\mathbf{x}+\mathbf{v}t,t) = R_f R R_f^{-1}\,\alpha_1(\mathbf{x},t) + (\delta_f R_f \mathbf{v} + \mathbf{k}_f - R_f R R_f^{-1} \mathbf{k}_f)\,\alpha_2(\mathbf{x},t)$$

The $\mathbb{R}$-linearity condition on $\alpha_f$ means that we may write

$$\alpha_2(\mathbf{x},t) = \alpha_2(\mathbf{x},0) + \alpha_2(\mathbf{0},t) \equiv \phi_1(\mathbf{x}) + \phi_2(t)$$

The homomorphism $\phi_1 \in \mathrm{Hom}\mathbb{R}(\mathbb{R}^3,\mathbb{R})$ is the zero homomorphism. We have the identity

$$\phi_1(R\mathbf{x}+\mathbf{v}t) \equiv \phi_1(\mathbf{x})$$

which by taking $R = e$ and $\mathbf{v} = -\mathbf{x}/t$ gives us

$$\phi_1(\mathbf{x}) = \phi_1(\mathbf{0}) = 0$$

It is clear that $\phi_2$ is a vector space isomorphism of $\mathbb{R}$ onto itself. Thus we may write

$$\phi_2(t) \equiv t.\phi_2(1) \equiv t\sigma_f \qquad (\sigma_f \neq 0)$$

(The real number $\sigma_f$ is non-zero since $\phi_2$ is a monomorphism.) Therefore we may write

$$\alpha_2(\mathbf{x}, t) = \alpha_2(\mathbf{0}, t) = \phi_2(t) = \sigma_f t, \qquad \sigma_f \neq 0$$

knowing $\alpha_2$ we may compute $\alpha_1$. For:

$$\alpha_1(R\mathbf{x} + \mathbf{v}t, t) = R_f R R_f^{-1} \alpha_1(\mathbf{x}, t) + (\delta_f R_f \mathbf{v} + k_f - R_f R R_f^{-1} k_f) \sigma_f t$$

If we take $R = e$, $\mathbf{x} = \mathbf{0}$ we obtain

$$\alpha_1(\mathbf{v}t, t) \equiv \alpha_1(\mathbf{v}t, 0) + \alpha_1(\mathbf{0}, t) = \delta_f R_f \sigma_f \mathbf{v}t + \alpha_1(\mathbf{0}, t)$$

Therefore we obtain:

$$\alpha_1(\mathbf{x}, 0) = \delta_f \sigma_f R_f \mathbf{x}$$

Define $\Phi_f \in \mathrm{Hom}(\mathbb{R}, \mathbb{R}^3)$ by $\Phi_f : t \mapsto \alpha_1(\mathbf{0}, t)$. Then $\Phi_f$ is $\mathbb{R}$-linear which implies

$$\Phi_f(t) \equiv t.\Phi_f(1) \equiv t\mathbf{k}_f^* \qquad \text{where } \mathbf{k}_f^* \in \mathbb{R}^3$$

But $\mathbf{k}_f^* = \sigma_f \mathbf{k}_f$. For, writing $\alpha_1(\mathbf{x}, t) = \sigma_f \delta_f R_f \mathbf{x} + \mathbf{k}_f^* t$ our identity on $\alpha_1$ implies

$$(\sigma_f \mathbf{k}_f - \mathbf{k}_f^*) = R_f R R_f^{-1}(\sigma_f \mathbf{k}_f - \mathbf{k}_f^*) \, \forall R \in O(3, \mathbb{R})$$

Taking $R = P$ we obtain $\sigma_f \mathbf{k}_f - \mathbf{k}_f^* = 0$. Consequently we have obtained the form below for $\alpha_f$

$$\alpha_f : (\mathbf{x}, t) \mapsto (\sigma_f \delta_f R_f \mathbf{x} + \sigma_f \mathbf{k}_f t, \sigma_f t)$$

where $\sigma_f$ and $\delta_f$ are non-zero scalars, $\mathbf{k}_f \in \mathbb{R}^3$ and $R_f \in SO(3, \mathbb{R})$. It remains to compute $\beta_f$. If we define functions $\beta_1$ and $\beta_2$ by

$$\beta_f \; (\mathbf{v}, R) \to (\beta_1(\mathbf{v}, R), \beta_2(\mathbf{v}, R))$$

The cocycle condition on $\beta$ yields the following indentities

$$\beta_2((\mathbf{v}_1, R_1)(\mathbf{v}_2, R_2)) = \beta_2(\mathbf{v}_1, R_1) + \beta_2(\mathbf{v}_2, R_2)$$

$$\beta_1((\mathbf{v}_1, R_1)(\mathbf{v}_2, R_2)) \\ = \beta_1(\mathbf{v}_1, R_1) + R_f R R_f^{-1} \beta_1(\mathbf{v}_2, R_2) + (\delta_f R_f \mathbf{v} + \mathbf{k}_f - R_f R R_f^{-1} \mathbf{k}_f) \beta_2(\mathbf{v}_2, R_2)$$

We have to know $\beta_2$ for a calculation of $\beta_1$. It is clear in fact that

$$\beta_2 = O(\beta_2 \in \mathrm{Hom}(E(3, \mathbb{R}), \mathbb{R}))$$

writing

$$\beta_1(\mathbf{v}, R) \equiv \beta_1(\mathbf{v}, e) + \beta_1(o, R) \equiv \rho_1(\mathbf{v}) + \rho_2(R)$$

we obtain the condition that $\rho_2 \in Z^1_{n \circ \gamma_f}(O(3,\mathbb{R}),\mathbb{R}^3)$ and $\rho_1(R\mathbf{v}) = R_f R R_f^{-1} \times \rho(v)$ for any $\mathbb{R} \in SO(3,R)$. We then obtain the solutions

$$\rho_2(R) = \mathbf{X}_f - R_f R R_f^{-1} \mathbf{X}_f \qquad \text{where } \mathbf{X}_f \in \mathbb{R}^3$$

$$\rho_1(\mathbf{v}) = -\tau_f R_f \mathbf{v} \qquad \text{where } \tau_f \text{ is } \textit{any} \text{ real number}$$

Therefore we obtain the following result

$$\beta_f: (\mathbf{v}, R) \mapsto (\mathbf{X}_f - R_f R R_f^{-1} \mathbf{X}_f - \tau_f R_f \mathbf{v}, 0)$$

and we may express any automorphism of $G$ by its action

$$f: ((\mathbf{x}, t), (\mathbf{v}, R)) \mapsto ((\sigma_f \delta_f R_f \mathbf{x} + \sigma_f \mathbf{k}_f t - \tau_f R_f \mathbf{v} + \mathbf{X}_f - R_f R R_f \mathbf{X}_f, \sigma_f t),$$
$$(\delta_f R_f \mathbf{v} + \mathbf{k}_f - R_f R R_f^{-1} \mathbf{k}_f, R_f R R^{-1}_f))$$

The correspondence $f \mapsto ((x_f, \tau_f), (k_f, R_f), (\delta_f . \sigma_f))$ is a group isomorphism:

$$\mathrm{Aut}(G) \cong G X_\omega(\mathbb{R}_m^* \times \mathbb{R}_m)$$

Where $\omega \in \mathrm{Hom}(\mathbb{R}_m^* \times \mathbb{R}_m, \mathrm{Aut}(G))$ is

$$\omega(\delta, \sigma): ((\mathbf{x}, t), (\mathbf{v}, R)) \to ((\delta\sigma\mathbf{x}, \sigma t), (\delta\mathbf{v}, R))$$

The canonical map $\mathrm{In}_G: G \mapsto \mathrm{Aut}(G)$ (where $\mathrm{In}(g)$ the inner automorphism induced by $g \in G$) is given by

$$\mathrm{In}: ((\mathbf{x}, t), (\mathbf{v}, R)) \mapsto ((\mathbf{x} - \mathbf{v}t, t), (\mathbf{v}, R), (\det(R), 1))$$

*Corollary* (1)

$\mathrm{Out}(G) \cong R_m^* \times R_m$ is a subgroup of $\mathrm{Aut}(G)$.

*Proof*

This is a trivial consequence of the fact that

$$\mathrm{Aut}(G) \cong G X_\omega(\mathbb{R}_m^* \times \mathbb{R}_m)$$

and $G/C(G) \cong \mathrm{Int}(G) \cong G$ since $C(G) = 1$.

*Corollary* (2)

$\mathrm{Aut}(\mathrm{Aut}(G)) \cong \mathrm{Aut}(G)$.

*Proof*

This is a trivial consequence of a Lemma (Michel, 1967) that if

$$\mathrm{Int}(\mathbf{G}) \lhd \mathrm{Aut}(\mathbf{G}), \qquad \mathrm{Aut}(\mathrm{Aut}(\mathbf{G})) \cong \mathrm{Aut}(\mathbf{G})$$

But $\mathrm{Int}(G) \cong G \lhd \mathrm{Aut}(G)$, since it is the only invariant subgroup of $\mathrm{Aut}(G)$ isomorphic to $G$.

*Corollary* (3)

All abstract automorphisms of $G$ are continuous if $G$ has the Lie group topology.

*Proof*

Trivially, from Theorem (1), describing the action of any automorphism of $G$, any automorphism is the product of a dilation, a rotation and a translation. Each of the constituent automorphisms is continuous.

## *Acknowledgements*

One of us (G.S.W.) acknowledges the receipt of an I.N.F.N. post-doctoral fellowship. We thank Professor Bruno Vitale for encouragement and support.

## *References*

Lévy-Leblond, J. M. (1971). Review Article on the Galilei Group. *Group Theory and its Applications*, Vol. II (Ed. Loebl). Academic Press.

Michel, L. (1966). An article in the lecture notes of the Brandeis Summer School 1965. *Axiomatic Field Theory* (Ed. M. Cretien and S. Dreser). Gordon and Breach.

Michel, L. (1967). 'Symmetries Internes et Invariance Relativiste.' An article in the lecture notes of the Summer School 1965. *Applications of Mathematics to Problems in Theoretical Physics* (Ed. E. Lurçat). Gordon and Breach.

Whiston, G. S. (1972). Associative and Non-Associative Automorphisms of Newtonian Space-Time. *International Journal of Theoretical Physics*, Vol. 6, No. 2, p. 79.